

SQUARING OPERATOR PÓLYA–SZEGÖ AND DIAZ–METCALF TYPE INEQUALITIES

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ABSTRACT. We square operator Pólya–Szegő and Diaz–Metcalf type inequalities as follows: If operator inequalities $0 < m_1^2 \leq A \leq M_1^2$ and $0 < m_2^2 \leq B \leq M_2^2$ hold for some positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$, then for every unital positive linear map Φ the following inequalities hold:

$$(\Phi(A)\sharp\Phi(B))^2 \leq \left(\frac{M_1M_2 + m_1m_2}{2\sqrt{M_1M_2m_1m_2}} \right)^4 \Phi(A\sharp B)^2$$

and

$$\left(\frac{M_2m_2}{M_1m_1} \Phi(A) + \Phi(B) \right)^2 \leq \left(\frac{(M_1m_1(M_2^2 + m_2^2) + M_2m_2(M_1^2 + m_1^2))^2}{8\sqrt{M_2M_1m_1m_2}M_1^2m_1^2M_2m_2} \right)^2 \Phi(A\sharp B)^2.$$

1. INTRODUCTION

Let $\mathbb{B}(\mathcal{H})$ denote the C^* -algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. Throughout the paper, a capital letter means an operator in $\mathbb{B}(\mathcal{H})$. If $\dim \mathcal{H} = n$, then $\mathbb{B}(\mathcal{H})$ can be identified with the space \mathbb{M}_n of all $n \times n$ complex matrices. We identify a scalar with the identity operator I multiplied by this scalar. An operator A is called positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$, and we then write $A \geq 0$. An operator A is said to be strictly positive (denoted by $A > 0$) if it is a positive invertible operator. For self-adjoint operators $A, B \in \mathbb{B}(\mathcal{H})$, we say $B \geq A$ if $B - A \geq 0$. For strictly positive operators, $A^2 \leq k^2 B^2$ for some constant k if and only if $(AB^{-1})^*(AB^{-1}) \leq k^2$ and this occurs if and only if $\|AB^{-1}\| \leq k$. A linear map $\Phi : \mathbb{B}(\mathcal{H}) \rightarrow \mathbb{B}(\mathcal{K})$ is called positive if $A \geq 0$ implies $\Phi(A) \geq 0$. If this implication holds for $>$ instead of \geq , we say that Φ is strictly positive. It is said to be unital if Φ preserves the identity operator. The operator norm is denoted by $\|\cdot\|$. For $A, B > 0$, the operator geometric mean $A\sharp B$ is defined by $A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$. Using a standard limit argument, this notion can be extended for positive operators A, B . The

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geometric mean operation is monotone, in the sense that $C_1 \leq D_1$ and $C_2 \leq D_2$ imply that $C_1 \sharp C_2 \leq D_1 \sharp D_2$.

Moslehian et al. [12, Theorem 2.1] gave operator Pólya–Szegő inequality (see also [8] for an interesting proof for matrices) and Diaz–Metcalf type inequality as follows:

Theorem 1.1. *Let Φ be a positive linear map. If $0 < m_1^2 \leq A \leq M_1^2$ and $0 < m_2^2 \leq B \leq M_2^2$ for some positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$,*

$$\Phi(A) \sharp \Phi(B) \leq \alpha \cdot \Phi(A \sharp B), \quad (1.1)$$

where

$$\alpha := \frac{1}{2} \left\{ \sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right\}.$$

Theorem 1.2. *Let Φ be a positive linear map. If $0 < m_1^2 \leq A \leq M_1^2$ and $0 < m_2^2 \leq B \leq M_2^2$ for some positive real numbers $m_1 \leq M_1$ and $0 < m_2 \leq M_2$, then the following inequality holds:*

$$\frac{M_2 m_2}{M_1 m_1} \Phi(A) + \Phi(B) \leq \left(\frac{M_2}{m_1} + \frac{m_2}{M_1} \right) \Phi(A \sharp B). \quad (1.2)$$

It is well known that t^s ($0 \leq s \leq 1$) is an operator monotone function and not so is t^2 ; see [13]. However, Fujii et al. [5, Theorem 6] use the Kantorovich inequality to show that t^2 is order preserving in a certain sense as follows:

Theorem 1.3. *Let $0 < m \leq A \leq M$ and $A \leq B$. Then*

$$A^2 \leq \frac{(M + m)^2}{4Mm} B^2.$$

Other similar results for the power p instead of 2 was given by Furuta in [6]. Lin [9] nicely reduced the study of squared operator inequalities to that of some norm inequalities, see also [10]. This paper intends to square the operator Pólya–Szegő inequality (1.1) and the Diaz–Metcalf type inequality (1.2) in different ways based on the above considerations.

2. RESULTS

We start our work with the following known result.

Lemma 2.1. [1] *Let Φ be any positive linear map and $A, B \geq 0$. Then*

$$\Phi(A \sharp B) \leq \Phi(A) \sharp \Phi(B). \quad (2.1)$$

Our first main result reads as follows.

Theorem 2.2. *Let Φ be a unital positive linear map. If $0 < m_1^2 \leq A \leq M_1^2$ and $0 < m_2^2 \leq B \leq M_2^2$ for some positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$, then the following inequality holds:*

$$\{\Phi(A)\sharp\Phi(B)\}^2 \leq \beta \cdot \Phi(A\sharp B)^2, \quad (2.2)$$

where

$$\beta := \begin{cases} \alpha^4 & \text{if } \alpha^2 \leq \sqrt{\frac{M_1 M_2}{m_1 m_2}}, \\ \sqrt{\frac{M_1 M_2}{m_1 m_2}} \left\{ 2\alpha^2 - \sqrt{\frac{M_1 M_2}{m_1 m_2}} \right\} & \text{if } \alpha^2 \geq \sqrt{\frac{M_1 M_2}{m_1 m_2}}, \end{cases} \quad (2.3)$$

and α is the number given in Theorem 1.1.

Proof. Put

$$M := M_1 M_2, \quad m := m_1 m_2, \quad C := \Phi(A\sharp B), \quad D := \Phi(A)\sharp\Phi(B)$$

and

$$\alpha = \frac{1}{2} \left(\frac{M + m}{\sqrt{Mm}} \right) \geq 1. \quad (2.4)$$

It follows from the monotone property of the operator geometric mean and

$$m_1^2 \leq A \leq M_1^2 \quad \text{and} \quad m_2^2 \leq B \leq M_2^2$$

that

$$\sqrt{m_1^2 m_2^2} \leq A\sharp B \leq \sqrt{M_1^2 M_2^2} \quad \text{and} \quad \sqrt{m_1^2 m_2^2} \leq \Phi(A)\sharp\Phi(B) \leq \sqrt{M_1^2 M_2^2}, \quad (2.5)$$

whence

$$0 < m \leq C \leq M \quad \text{and} \quad 0 < m \leq D \leq M \quad (2.6)$$

Theorem 1.1 and inequality (2.1) yield that

$$C \leq D \leq \alpha C. \quad (2.7)$$

From (2.6) we have $(M - C)(C - m) \geq 0$ and $(M - D)(D - m) \geq 0$. Hence

$$C^2 \leq (M + m)C - Mm \quad \text{and} \quad D^2 \leq (M + m)D - Mm. \quad (2.8)$$

Employing (2.7) and (2.8) we have

$$0 \leq C^{-1} D^2 C^{-1} \leq \{\alpha(M + m)C - Mm\} C^{-2}. \quad (2.9)$$

Consider the real function $f(t)$ on $(0, \infty)$ defined as

$$f(t) := \frac{\alpha(M+m)t - Mm}{t^2}. \quad (2.10)$$

Then we can conclude from (2.6), (2.9) and (2.10) that

$$C^{-1}D^2C^{-1} \leq \max_{m \leq t \leq M} f(t). \quad (2.11)$$

Notice that

$$f(m) = \frac{\alpha(M+m) - M}{m} \geq \frac{\alpha(M+m) - m}{M} = f(M)$$

and

$$f'(t) = \frac{2Mm - \alpha(M+m)t}{t^3}. \quad (2.12)$$

The function $f(t)$ has only one stationary (= maximum) point at

$$t_0 := \frac{2Mm}{\alpha(M+m)} \quad (2.13)$$

with the maximum value (by (2.4))

$$f(t_0) = \frac{\alpha^2(M+m)^2}{4Mm} = \alpha^4. \quad (2.14)$$

Therefore we can conclude that

$$\max_{m \leq t \leq M} f(t) \leq \begin{cases} f(t_0) & \text{if } m \leq t_0 \\ f(m) & \text{if } m \geq t_0. \end{cases}$$

It is immediate to see from (2.4) and (2.13) that $m \leq t_0$ is equivalent to $\alpha^2 \leq \sqrt{\frac{M}{m}}$.

Finally we have again by (2.4)

$$f(m) = 2\alpha^2 \sqrt{\frac{M}{m}} - \frac{M}{m} = \sqrt{\frac{M}{m}} \left\{ 2\alpha^2 - \sqrt{\frac{M}{m}} \right\}. \quad (2.15)$$

This completes the proof of Theorem □

Remark 2.3. The axiomatic theory for operator means of positive invertible operators have been developed by Kubo and Ando [7]. There exists an affine order isomorphism between the class of operator means σ and the class of positive operator monotone functions f defined on $(0, \infty)$ via $f(t)I = I\sigma(tI)$ ($t > 0$) and $A\sigma B = A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$. The function $f_{\sharp\mu}(t) = t^\mu$ on $(0, \infty)$ for $\mu \in (0, 1)$ gives the operator weighted geometric mean $A\sharp_\mu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\mu A^{\frac{1}{2}}$. The case $\mu = 1/2$ gives rise to the geometric mean $A\sharp B$. It should be noted that

$$\Phi(A\sigma B) \leq \Phi(A)\sigma\Phi(B) \quad (2.16)$$

is known as an Ando type inequality in the literature, see [1]. We need the next result appeared in [13] in some general forms.

Theorem 2.4. Let Φ be a unital positive linear map and σ be an operator mean with the representing function f . If $0 < m_1^2 \leq A \leq M_1^2$ and $0 < m_2^2 \leq B \leq M_2^2$ for some positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$, then

$$\Phi(A)\sigma\Phi(B) \leq \alpha\Phi(A\sigma B) \quad (2.17)$$

with

$$\alpha = \max \left\{ \frac{f(t)}{\mu_f t + \nu_f} : \frac{m_2^2}{M_1^2} \leq t \leq \frac{M_2^2}{m_1^2} \right\}, \quad (2.18)$$

where

$$\mu_f = \frac{f(M_2^2/m_1^2) - f(m_2^2/M_1^2)}{(M_2^2/m_1^2) - (m_2^2/M_1^2)} \quad \text{and} \quad \nu_f = \frac{(M_2^2/m_1^2)f(m_2^2/M_1^2) - (m_2^2/M_1^2)f(M_2^2/m_1^2)}{(M_2^2/m_1^2) - (m_2^2/M_1^2)}.$$

If $\sigma = \sharp_\mu$ ($\mu \in [0, 1]$), then

$$\alpha = \frac{\mu^\mu ((M_2^2/m_1^2) - (m_2^2/M_1^2)) ((M_2^2/m_1^2)(m_2^2/M_1^2)^\mu - (m_2^2/M_1^2)(M_2^2/m_1^2)^\mu)^{1-\mu}}{(1-\mu)^{\mu-1} ((M_2^2/m_1^2)^\mu - (m_2^2/M_1^2)^\mu)^\mu},$$

see [14, Theorem 3]. In particular, for $\sigma = \sharp$ we get the operator Pólya–Szegő inequality. It is easy to see that, by utilizing the same argument as in Theorem 2.2, the following general but complicated form of Theorem 2.2 holds:

Theorem 2.5. Under the same conditions as in Theorem 2.2

$$(\Phi(A)\sigma\Phi(B))^2 \leq \beta \cdot \Phi(A\sigma B)^2,$$

where

$$\beta = \max_{m_1^2 f(m_1^{-2} m_2^2) \leq t \leq M_1^2 f(M_1^{-2} M_2^2)} \frac{\alpha(M_1^2 f(M_1^{-2} M_2^2) + m_1^2 f(m_1^{-2} m_2^2))t - m_1^2 M_1^2 f(m_1^{-2} m_2^2) f(M_1^{-2} M_2^2)}{t^2}.$$

Let us return to the case of usual operator geometric mean \sharp . An immediate consequence of Theorem 2.2 reads as follows. It can be also deduced from Theorems 1.1 and 1.3.

Corollary 2.6. Under the same conditions as in Theorem 2.2,

$$\{\Phi(A)\sharp\Phi(B)\}^2 \leq \alpha^4 \cdot \Phi(A\sharp B)^2$$

Remark 2.7. It is easy to see that the coefficient α^2 from (1.1) is smaller than β in (2.2), but we obtained the relation between $(\Phi(A)\sharp\Phi(B))^2$ and $\Phi(A\sharp B)^2$.

The following consequence may be regarded as a Grüss type inequality, see [11] and references therein.

Corollary 2.8. *Let Φ be a unital positive linear map. If $0 < m_1^2 \leq A \leq M_1^2$ and $0 < m_2^2 \leq B \leq M_2^2$ for some positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$, then the following inequality holds:*

$$(\Phi(A)\sharp\Phi(B))^2 - \Phi(A\sharp B)^2 \leq (\beta - 1) M_1^2 M_2^2 \quad (2.19)$$

where β is defined by (2.3).

Proof. It follows from (2.2) that

$$(\Phi(A)\sharp\Phi(B))^2 - \Phi(A\sharp B)^2 \leq (\beta - 1) \Phi(A\sharp B)^2. \quad (2.20)$$

It follows from (2.5) that

$$m_1^2 m_2^2 \leq \Phi(A\sharp B)^2 \leq M_1^2 M_2^2. \quad (2.21)$$

Employing (2.20) and (2.21) we infer (2.19). \square

To achieve our next result we need some auxiliary lemmas. The first Lemma is a consequence of the Jensen inequality and the operator convexity of $f(t) = 1/t$.

Lemma 2.9. [3] *Let Φ be a unital strictly positive linear map and $A > 0$. Then*

$$\Phi(A)^{-1} \leq \Phi(A^{-1}). \quad (2.22)$$

The next lemma is proved for matrices but a careful investigation shows that it is true for operators on an arbitrary Hilbert space.

Lemma 2.10. [2] *Let $A, B \geq 0$. Then*

$$\|AB\| \leq \frac{1}{4} \|A + B\|^2. \quad (2.23)$$

Remark 2.11. Lemma 2.10 is due to Bhatia and Kittaneh in [2, Theorem 1]. They proved the result for the finite dimensional case. However, for infinite dimensional case, the result for operator norm is also true. Also, we notice that if A, B are compact operators, then a stronger result can be found in [4].

The following lemma includes the well-known operator geometric-arithmetic mean inequality.

Lemma 2.12. [13, Theorem 1.27] *Let $A, B \geq 0$. Then*

$$A\sharp B \leq \frac{A+B}{2}. \quad (2.24)$$

The definition of operator geometric mean easily yields the next lemma.

Lemma 2.13. *Let $A, B > 0$. Then*

$$(A\sharp B)^{-1} = A^{-1}\sharp B^{-1}. \quad (2.25)$$

We now present our next main result.

Theorem 2.14. *Let Φ be a unital positive linear map. If $0 < m_1^2 \leq A \leq M_1^2$ and $0 < m_2^2 \leq B \leq M_2^2$ for some positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$, then the following inequality holds:*

$$\begin{aligned} & \left(\frac{M_2 m_2}{M_1 m_1} \Phi(A) + \Phi(B) \right)^2 \\ & \leq \left(\frac{(M_1 m_1 (M_2^2 + m_2^2) + M_2 m_2 (M_1^2 + m_1^2))^2}{8 \sqrt{M_2 M_1 m_1 m_2} M_1^2 m_1^2 M_2 m_2} \right)^2 \Phi(A\sharp B)^2. \end{aligned} \quad (2.26)$$

Proof. It is easy to see that (2.26) is equivalent to

$$\left\| \left(\frac{M_2 m_2}{M_1 m_1} \Phi(A) + \Phi(B) \right) \Phi(A\sharp B)^{-1} \right\| \leq \frac{(M_1 m_1 (M_2^2 + m_2^2) + M_2 m_2 (M_1^2 + m_1^2))^2}{8 \sqrt{M_2 M_1 m_1 m_2} M_1^2 m_1^2 M_2 m_2}.$$

Evidently

$$(M_1^2 - A)(m_1^2 - A)A^{-1} = A + m_1^2 M_1^2 A^{-1} - m_1^2 - M_1^2 \leq 0,$$

and hence

$$m_1^2 M_1^2 \Phi(A^{-1}) + \Phi(A) \leq M_1^2 + m_1^2,$$

which implies that

$$M_2 m_2 m_1 M_1 \Phi(A^{-1}) + \frac{M_2 m_2}{M_1 m_1} \Phi(A) \leq \frac{M_2 m_2}{M_1 m_1} (M_1^2 + m_1^2). \quad (2.27)$$

In the same way, we have

$$m_2^2 M_2^2 \Phi(B^{-1}) + \Phi(B) \leq M_2^2 + m_2^2. \quad (2.28)$$

Inequalities (2.27) and (2.28) yield that

$$\begin{aligned}
& \frac{M_2 m_2}{M_1 m_1} (M_1^2 + m_1^2) + M_2^2 + m_2^2 \\
& \geq M_2 m_2 m_1 M_1 \Phi(A^{-1}) + m_2^2 M_2^2 \Phi(B^{-1}) + \Phi(B) + \frac{M_2 m_2}{M_1 m_1} \Phi(A) \\
& \geq 2\sqrt{M_2 M_1 m_1 m_2} M_2 m_2 (\Phi(A^{-1}) \sharp \Phi(B^{-1})) + (\Phi(B) + \frac{M_2 m_2}{M_1 m_1} \Phi(A)) \quad (\text{by (2.24)}) \\
& \geq 2\sqrt{M_2 M_1 m_1 m_2} M_2 m_2 \Phi(A^{-1} \sharp B^{-1}) + (\Phi(B) + \frac{M_2 m_2}{M_1 m_1} \Phi(A)) \quad (\text{by (2.1)}) \\
& \geq 2\sqrt{M_2 M_1 m_1 m_2} M_2 m_2 \Phi((A \sharp B)^{-1}) + (\Phi(B) + \frac{M_2 m_2}{M_1 m_1} \Phi(A)) \quad (\text{by (2.25)}) \\
& \geq 2\sqrt{M_2 M_1 m_1 m_2} M_2 m_2 \Phi(A \sharp B)^{-1} + (\Phi(B) + \frac{M_2 m_2}{M_1 m_1} \Phi(A)) \quad (\text{by (2.22)})
\end{aligned}$$

which implies that

$$\begin{aligned}
& 2\sqrt{M_2 M_1 m_1 m_2} M_2 m_2 \Phi(A \sharp B)^{-1} + (\Phi(B) + \frac{M_2 m_2}{M_1 m_1} \Phi(A)) \\
& \leq \frac{M_2 m_2}{M_1 m_1} (M_1^2 + m_1^2) + M_2^2 + m_2^2.
\end{aligned} \tag{2.29}$$

Now we have

$$\begin{aligned}
& \left\| \left(\frac{M_2 m_2}{M_1 m_1} \Phi(A) + \Phi(B) \right) \left(2\sqrt{M_2 M_1 m_1 m_2} M_2 m_2 \Phi(A \sharp B)^{-1} \right) \right\| \\
& \leq \frac{1}{4} \left\| \left(\frac{M_2 m_2}{M_1 m_1} \Phi(A) + \Phi(B) \right) + (2\sqrt{M_2 M_1 m_1 m_2} M_2 m_2) \Phi(A \sharp B)^{-1} \right\|^2 \quad (\text{by (2.23)}) \\
& \leq \frac{(M_1 m_1 (M_2^2 + m_2^2) + M_2 m_2 (M_1^2 + m_1^2))^2}{4M_1^2 m_1^2} \quad (\text{by (2.29)})
\end{aligned}$$

which is equivalent to

$$\left\| \left(\frac{M_2 m_2}{M_1 m_1} \Phi(A) + \Phi(B) \right) \Phi(A \sharp B)^{-1} \right\| \leq \frac{(M_1 m_1 (M_2^2 + m_2^2) + M_2 m_2 (M_1^2 + m_1^2))^2}{8\sqrt{M_2 M_1 m_1 m_2} M_1^2 m_1^2 M_2 m_2}.$$

□

Remark 2.15. It is easy to obtain that

$$\frac{M_2}{m_1} + \frac{m_2}{M_1}$$

in (1.2) is smaller than

$$\frac{(M_1 m_1 (M_2^2 + m_2^2) + M_2 m_2 (M_1^2 + m_1^2))^2}{8\sqrt{M_2 M_1 m_1 m_2} M_1^2 m_1^2 M_2 m_2}$$

in (2.26), but we obtain the relation between $\left(\frac{M_2 m_2}{M_1 m_1} \Phi(A) + \Phi(B) \right)^2$ and $\Phi(A \sharp B)^2$.

Conjecture 2.16. Let Φ be a unital positive linear map. If $0 < m_1^2 \leq A \leq M_1^2$ and $0 < m_2^2 \leq B \leq M_2^2$ for some positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$, then the following inequality hold:

$$(\Phi(A) \sharp \Phi(B))^2 \leq \frac{1}{4} \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2 \Phi(A \sharp B)^2$$

and

$$\left(\frac{M_2 m_2}{M_1 m_1} \Phi(A) + \Phi(B) \right)^2 \leq \left(\frac{M_2}{m_1} + \frac{m_2}{M_1} \right)^2 \Phi(A \sharp B)^2.$$

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